Mathematica Moravica
Vol. **9** (2005), 17–20

ON THE p-REDUCED ENERGY OF A GRAPH

MIRJANA LAZIĆ

Abstract. Let G be a simple connected graph of order n and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the spectrum of G. Then the sum $S_k^l(G) = |\lambda_{k+1}| + |\lambda_{k+2}| + \cdots + |\lambda_{n-l}|$ is called (k,l)-reduced energy of G, where k, l are two fixed nonnegative integers [2]. In this work, we make a generalization of the (k,l)-reduced energy, as follows: for any fixed $p \in N$, the sum $S_k^l(G,p) = |\lambda_{k+1}|^p + |\lambda_{k+2}|^p + \cdots + |\lambda_{n-l}|^p$ is called the p-th (k,l)-reduced energy of the graph G. We also here introduce definitions of some other kinds of the p-reduced energies and we prove some properties of them.

1. Introduction

In this paper we consider only simple connected graphs. The vertex set of a graph G is denoted by $V(G)$, and its order by $|G|$. The spectrum of such a graph is the set $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of eigenvalues of its 0-1 adjacency matrix [1].

Let N_0 be the set of all nonnegative integers and $l \in N_0$ be a fixed number. For any graph G with $|G| = n > l$ the sum of eigenvalues $|\lambda_1| + |\lambda_2| + \cdots + |\lambda_{n-l}|$ is denoted by $S_+^l(G)$ and is called l-positive reduced energy of G [2]. In this work we shall define the sum $S_+^l(G,p) = |\lambda_1|^p + |\lambda_2|^p + \cdots + |\lambda_{n-l}|^p$ which is called the p-th l-positive reduced energy of the graph G. It contains at least the largest eigenvalue $\lambda_1(G)$ of G. We note that $|\lambda_1| \geq 1$, hence $S_+^l(G,p) \geq 1$ for any graph G . For any real $a \geq 1$ and any $l \in N_0$ and $p \in N$, we can consider the class of graphs $E_+^l(p,a) = \{G : S_+^l(G,p) \leq a\}$.

Now, we prove an important property of the general class $E_+^l(p,a)$.

Theorem 1. *For every constant $a \geq 1$ and for any fixed $l \in N_0$ and $p \in N$, the class of connected graphs $E_+^l(p,a) = \{G : |\lambda_1|^p + \cdots + |\lambda_{n-l}|^p \leq a\}$ is finite.*

Proof. Let a be any real number ($a \geq 1$) and l be nonnegative integer. Let G be a graph of order $n > l$ from the class $E_+^l(p,a)$. Then

$$S_+^l(G,p) = |\lambda_1|^p + \cdots + |\lambda_{n-l}|^p \leq a, \tag{1}$$

2000 *Mathematics Subject Classification.* Primary: 05C50.
Key words and phrases. Simple graphs, Energy of the graph.

which provides that

$$\sum_{|\lambda_i|\geq 1}|\lambda_i|^p \leq \sum_{i=1}^{n-l}|\lambda_i|^p + \sum_{i=n-l+1}^{n}|\lambda_i|^p \leq \tag{2}$$

$$\leq a + \sum_{i=n-l+1}^{n}|\lambda_1|^p = a + l\cdot|\lambda_1|^p \leq a(l+1)$$

Relations (1) and (2) now give

$$2(n-1) \leq 2m = \sum_{i=1}^{n}|\lambda_i|^2 \leq \sum_{|\lambda_i|<1}|\lambda_i| + \sum_{|\lambda_i|\geq 1}|\lambda_i|^p \leq (n-1)\cdot 1 + a(l+1) \tag{3}$$

where m is the number of edges of the graph G. Using (3) we find that $n \leq a(l+1)+1$, which completes the proof. □

Next, let $k, l \in N_0$ and $p \in N$ be any fixed numbers. For any graph G with $|G| = n > k+l$, the sum of eigenvalues $|\lambda_{k+1}| + |\lambda_{k+2}| + \cdots + |\lambda_{n-l}|$ is denoted by S_k^l and is called (k,l)-reduced energy of G. The sum of p-th degrees of eigenvalues $|\lambda_{k+1}|^p + |\lambda_{k+2}|^p + \cdots + |\lambda_{n-l}|^p$ is denoted by $S_k^l(G,p)$ and is called the p-th (k,l)-reduced energy of the graph G. For any real $a > 0$, for any $k, l \in N_0$, and for $p \in N$, we can consider the class of graphs

$$E_k^l(p,a) = \{G : S_k^l(G,p) \leq a\}.$$

We note that the p-th $(0,k)$-reduced energy of G is p-th k-positive reduced energy of G. We can always assume that $k, l \in N_0$ and $p \in N$.

Since the complete bipartite graph K_m^n belongs to the class $E_k^l(p,a)$ for any $m, n \in N$, the class $E_k^l(p,a)$ is always infinite. In what follows, we will prove an important property of this kind of energy on the set so-called canonical graphs.

We say that two vertices $x, y \in V(G)$ are equivalent in G and write $x \sim y$ if x is non-adjacent to y, and x and y have exactly the same neighbors in G. Relation $\sim$ is an equivalence relation on the vertex set $V(G)$. The corresponding quotient graph is denoted by $\widetilde{G}$, and is called the canonical graph of G.

Consequently, for any real $a > 0$ and $k, l, p \in N$, we can consider the class of the corresponding canonical graphs

$\tilde{E}_k^l(p,a) = \{G : G \in E_k^l(p,a)$ is a canonical graph$\}$.

If $k = l$, then $S_k^k(p,a)$, $E_k^k(p,a)$, $\tilde{E}_k^k(p,a)$ are simply denoted by $S_k(p,a)$, $E_k(p,a)$ and $\tilde{E}_k(p,a)$, respectively.

We now prove an important property of the class $\tilde{E}_k(p,a)$ $\quad (a > 0, k, p \in N)$. It is based on two theorems proved in [3].

Theorem 2. *For every constant $a > 0$ and any positive integers $k, p \in N$, the class of connected graphs $\tilde{E}_k(p,a)$ is finite.*

Proof. On the contrary, we shall suppose that for some $a > 0$ (a is a positive integer) and $k, p \in N$ the set $\tilde{E}_k(p,a)$ is infinite. By Theorem proved in [3], for any real number $A > 0$, there exists a graph $G \in \tilde{E}_k(p, a)$, which has $q > A$ nonzero eigenvalues. This graph will satisfy the relation

$$|\lambda_{k+1}|^p + |\lambda_{k+2}|^p + \cdots + |\lambda_{n-k}|^p \leq a. \tag{4}$$

Suppose that $\lambda_r > \lambda_{r+1} = \cdots = \lambda_{r+s} = 0 > \lambda_{r+s+1}$, where $s = n - q$ is the multiplicity of zero of this graph. The characteristic polynomial of G is then

$$P_n(\lambda) = \lambda^s(\lambda^q + a_1\lambda^{q-1} + \cdots + a_q),$$

where $|a_q| = \lambda_1 \cdots \lambda_r \cdot |\lambda_{r+s+1}| \cdots |\lambda_n|$.

By Theorem also proved in [3], we shall suppose that $s = 0$ and thus $n = q$. Also, we can assume that n is so large that we have $\sqrt{n} \geq a + 6k$.

It is clear that $|\lambda_i| \leq n - 1$ for $i \in \{1, 2, \ldots, k\} \cup \{n - k + 1, n - k + 2, \ldots, n\}$, and $|\lambda_i| \leq \sqrt{n}$ for $i = k + 1, k + 2, \ldots, n - k$.

We can assign with t the total number of eigenvalues λ_i, with $|\lambda_i| \leq 1/\sqrt{n}$ $(i = k+1, k+2, \ldots, n-k)$. It is easy to see that $t > a+4k$. On the contrary, we would have that there exist at least $n-(2k+t)$ eigenvalues λ_i $(k+1 \leq i \leq n-k)$ with $|\lambda_i| > 1/\sqrt{n}$. By relation (4) we have

$$a \geq \sum_{i=k+1}^{n-k} |\lambda_i|^p > \frac{n - (2k + t)}{\sqrt{n}} > \sqrt{n} - \frac{a + 6k}{\sqrt{n}}. \tag{5}$$

From relation $\sqrt{n} \geq a + 6k$ and relation (5) we have $a > \sqrt{n} - 1$ what is a contradiction.

Denote with t_0 the total number of all eigenvalues $\lambda_i (i = k+1, k+2, \ldots, n-k)$ with $|\lambda_i| > 1$. By relation (4) we have

$$a \geq |\lambda_{k+1}|^p + \cdots + |\lambda_{n-k}|^p \geq \sum_{|\lambda_i|>1} |\lambda_i|^p \geq t_0, \tag{6}$$

which provides that $t_0 \leq a$.

We now have

$$\begin{aligned} |a_n| &= (|\lambda_1| \cdots |\lambda_k|)(|\lambda_{k+1}| \cdots |\lambda_{n-k}|)(|\lambda_{n-k+1}| \cdots |\lambda_n|) \leq \\ &\leq (n-1)^{2k} \underbrace{\sqrt{n} \cdot \sqrt{n} \cdots \sqrt{n}}_{t_0} \cdot \underbrace{\frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{n}} \cdots \frac{1}{\sqrt{n}}}_{t} \cdot \underbrace{1 \cdot 1 \cdots 1}_{n-(t+t_0+2k)} < 1, \end{aligned}$$

which is a contradiction ($|a_n| \in N \quad a_n \neq 0$). Consequently, the set $\tilde{E}_k(p, a)$ is finite for every $a > 0$ and every $k, p \in N$. □

Corollary 1. *For every constant $a > 0$ and any positive integers $k, l, p \in N$, the class of connected graphs $\tilde{E}_k^l(p, a)$ is finite.*

Proof. Without loss of generality, we can assume that $k \geq l$. Let G be any graph from the class $\tilde{E}_k^l(p, a)$. Since

$$a \geq \sum_{i=k+1}^{n-l} |\lambda_i|^p = \sum_{i=k+1}^{n-k} |\lambda_i|^p + \sum_{i=n-k+1}^{n-l} |\lambda_i|^p \geq \sum_{i=k+1}^{n-k} |\lambda_i|^p$$

we have $G \in \tilde{E}_k(p,a)$, thus $\tilde{E}^l_k(p,a) \subseteq \tilde{E}_k(p,a)$. Since the class $\tilde{E}_k(p,a)$ is finite for every $a > 0$ and every $k, p \in N$, we get the statement. □

References

[1] D.Cvetković, M.Doob and H.Sachs, *Spectra of Graphs-Theory and Application*, Deutscher Verlag der Wissenschaften, Berlin and Academic Press, NewYork, 1982.

[2] M.Lepović, *Some kinds of energies of graphs*, Discrete Mathematics **128**(1994), 277-282.

[3] A.Torgašev, *On infinite graphs with three and four non-zero eigenvalues*, Bull. Serb. Acad. Sci. et Arts LXIV (Sci. Math.) **11**(1981), 39-48.

Faculty of Science
Institute of Mathematics and Informatics
P.O.Box 60, Radoja Domanovića 12
34000 Kragujevac
Serbia and Montenegro
E-mail address: mmmvl@kg.ac.yu